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Non-commutative soliton scattering

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Abstract: We study solitons in three-dimensional non-commutative scalar field theory at infinite non-commutativity parameter θ. We find the metric on the relative moduli space of all solitons of the form $|n\rangle\langle n|$ and show that it is Kähler. We then find the geodesics of this metric and study the scattering of these solitons. In particular we find that the scattering is generally right angle for small values of the impact parameter. We can understand this behaviour in terms of a conical singularity at the origin of moduli space.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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1. Introduction

Quantum field theories on non-commutative spaces have lately received a revival of interest, both as seemingly consistent non-local deformations of the highly constrained structure of local quantum field theory, and as theories that appear in various limits of M-theory compactifications [1] or the low-energy effective theory of D-branes in the presence of a background Neveu-Schwarz B-field [2, 3]. Conversely, a study of the perturbative properties of non-commutative field theory reveals that it has many stringy features [4]–[7].

Recently, following the pioneering work [8], soliton solutions of non-commutative field theory have been studied [9]–[15]. In particular it seems that the solutions often have nice interpretation as string theory states. Particularly interesting is the case when the non-commutative solitons are combined with tachyon condensation [9, 10, 15]. Then one can study various string states in the standard closed string vacuum without D-branes. The trick used is to start with a D-brane anti-D-brane pair with a NS B-field turned on; after the tachyon field condenses into a soliton configuration, the brane anti-brane pair annihilate each other so that only the soliton remains.

This paper is organized as follows. In section 2 we review the results of [8]. In section 3 we study the simplest two soliton solution in more detail. In section 4 we find the metric on the relative moduli space of the simplest two soliton solution and find the geodesics to be able to study the scattering. We find that the metric is Kähler with a conical singularity in the center which gives rise to right angle scattering for small impact parameters. In section 5 we find the metric on the relative moduli

space of soliton solutions of the type $|n\rangle\langle n|$ for arbitrary n. We give the general expression for the metric which is again Kähler with the same qualitative behavior for the scattering. Finally we end with a discussion in section 6.

2. Non-commutative solitons

In this paper we study three-dimensional scalar field theory in a space time where the spacelike coordinates $\hat{x}_1, \hat{x}_2$ are non-commutative. The energy functional is

$$E = \frac{1}{2g^2} \int d^2x \left(\partial_t\phi\partial_t\phi + \partial_1\phi\partial_1\phi + \partial_2\phi\partial_2\phi + V(\phi)\right), \tag{2.1}$$

where all fields are multiplied using the non-local star product,

$$A \star B = e^{i\frac{\theta}{2}(\partial_1\partial'_2 - \partial_2\partial'_1)} A(x)\, B(x')\,|_{x=x'}\,. \tag{2.2}$$

We are interested in the limit where $\theta \to \infty$ and to study that limit we rescale the non-commutative coordinates $x \to x\sqrt{\theta}$. Then the star product itself becomes independent of θ but we get explicit factors of θ in the energy functional in front of the kinetic term and the potential term $V(\phi)$. Thus, in the limit $\theta \to \infty$, we get the energy functional

$$E = \frac{\theta}{2g^2} \int d^2x \left(\partial_t\phi\partial_t\phi + V(\phi)\right). \tag{2.3}$$

If we are interested in time independent solutions we can drop the time derivatives and the minimal energy solutions fulfill the equation

$$\frac{\partial V}{\partial \phi} = 0\,. \tag{2.4}$$

In [8] it was shown that any field satisfying the relation

$$\phi \star \phi = \phi\,, \tag{2.5}$$

gives a solution $\lambda\phi$, where λ is an extremum of the *function* $V(x)$. By introducing the creation and annihilation operators

$$a = \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2}}\,; \qquad a^\dagger = \frac{\hat{x}_1 - i\hat{x}_2}{\sqrt{2}}\,, \tag{2.6}$$

they were able to reformulate the problem in terms of the familiar operators and states of the harmonic oscillator. Namely, any function of $\hat{x}_1, \hat{x}_2$ can be written as a function of $a, a^\dagger$ which, as an operator, can be rewritten in terms of the operators

$$|m\rangle\langle n| = :\frac{a^{\dagger m}}{\sqrt{m!}}\, e^{-a^\dagger a} \frac{a^n}{\sqrt{n!}}:\,, \tag{2.7}$$

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where the double dots denote normal ordering. In particular, any function satisfying relation (2.5) is a projection operator and can alway be written as a sum of the form $\sum_i |A_i\rangle\langle A_i|$ for arbitrary orthonormal states $|A_i\rangle$.

Using this method, taking into account that the $|m\rangle\langle n|$ operators are normal ordered and the $\phi(\hat{x})$ operators are Weyl ordered, the authors of [8] were able to find a whole set of radially symmetric solutions ϕ_n corresponding to the operators $|n\rangle\langle n|$,

$$\phi_n = 2(-1)^n e^{-r^2} L_n\left(2r^2\right), \tag{2.8}$$

where $r^2 = x_1^2 + x_2^2$ and L_n are the Laguerre polynomials. These are all blobs centered at the origin $r = 0$.

The energy functional is invariant under unitary transformations of the operators

$$|m\rangle\langle n| \longrightarrow U|m\rangle\langle n|U^\dagger . \tag{2.9}$$

One particularly interesting unitary operator is $U = e^{a^\dagger z - a\bar{z}}$ which acts as a translation operator. Acting with this U on any of the above states simply translates the state to be centered around the point $z = \frac{1}{\sqrt{2}}(z_1 + iz_2)$. We use this in the next section to make solitons that can move around and scatter off each other.

3. The two soliton solution

We want to study solutions corresponding to two of the basic solitons in the previous section. By making the positions of these solitons weakly time dependent (the adiabatic approximation), we can derive a metric on the relative moduli space and study scattering of these solitons. We begin with the $|0\rangle\langle 0|$ state. We thus would like to find a solution to (2.5) which, at large separation of the solitons should look like

$$\phi_0(x - z) + \phi_0(x + z), \tag{3.1}$$

where the relative distance of the solitons is $2z$. From the previous section we know that

$$\phi_0(x - z) \propto U|0\rangle\langle 0|U^\dagger , \tag{3.2}$$

where $U = e^{a^\dagger z - a\bar{z}}$. Since U creates the usual coherent state $U|0\rangle = e^{-|z|^2/2} e^{a^\dagger z}|0\rangle = |z\rangle$, the solution we are looking for can, for large z, be written as

$$|z\rangle\langle z| + |-z\rangle\langle -z| , \tag{3.3}$$

but unfortunately this is not a good solution for small values of z since the states $|\pm z\rangle$ are not orthogonal. Therefore, the authors of [8] defined new, mutually orthogonal, states

$$|z_\pm\rangle = \frac{|z\rangle \pm |-z\rangle}{\sqrt{2\left(1 \pm e^{-2|z|^2}\right)}} , \tag{3.4}$$

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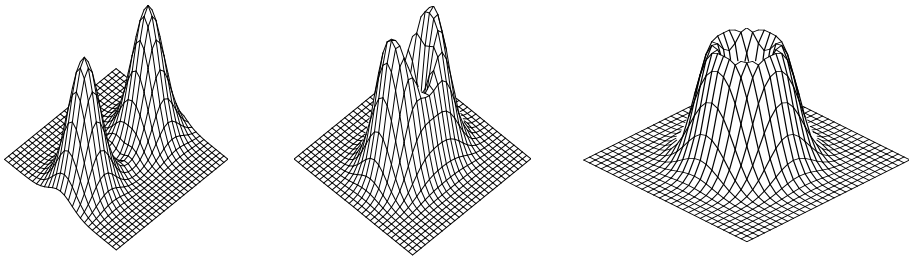

Figure 1: The 2 soliton solution for $z = 2$, 0.9 and 0, respectively.

in terms of which we can write the solution as

$$\lambda \left(|z_+\rangle\langle z_+| + |z_-\rangle\langle z_-| \right), \tag{3.5}$$

where λ is an appropriate normalization. Transforming this operator into a wave function we get

$$\frac{\phi_0 (r - z) + \phi_0 (r + z) - 2e^{-2|z|^2} \phi_0 (r) \cos (2r \wedge z)}{1 - e^{-4|z|^2}}, \tag{3.6}$$

where $r \wedge z = x_1 z_2 - x_2 z_1$ is a non-commutative factor reminiscent of the phases that appear in the star product. Note that for large separation $z \to \infty$, the total wave function becomes that of two ϕ_0 solitons at positions $\pm z$, whereas when the solitons come very close to each other, $z \to 0$, the solution goes to $2r^2 \phi_0 (r)$ which does not look like two separate ϕ_0 solutions, but rather like a new object, a "charge two" solution (actually, it is a superposition of ϕ_1 and ϕ_0). This is exactly what happens for ordinary monopoles [16]; at large distance one can see two distinct monopoles but when they get close together they lose their separate identity and merge into a charge two monopole. To illustrate this we have plotted the two soliton solution (3.6) for various values of the separation in figure 1.

4. Scattering — the simple case

Now consider the two soliton solution (3.6) from the previous section and let the solitons move by making the parameter z time-dependent. If the solitons move slowly enough, we can make the approximation that the potential energy of the energy functional (2.1) is always at the minimum and constant. However, we get a new contribution from the time derivatives in (2.1)[1]

$$E = \frac{\theta}{g^2} \int d^2x \left(2\partial_z \phi \partial_{\bar{z}} \phi \dot{z} \dot{\bar{z}} + \partial_z \phi \partial_z \phi \dot{z} \dot{z} + \partial_{\bar{z}} \phi \partial_{\bar{z}} \phi \dot{\bar{z}} \dot{\bar{z}} \right). \tag{4.1}$$

[1]This analysis is in the spirit of the discussion of monopole scattering in [18, 19] and vortex scattering in [20, 21]. A metric on quantum states was introduces in [17].

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The coefficients of the time derivatives of the z parameters are the components of the metric of the relative moduli space and the equation of motion *for the z's* is the geodesic equation in this metric. We thus find that the metric on the relative moduli space of these solitons is (up to a factor)

$$\begin{aligned} g_{z\bar{z}} &= \int d^2x \partial_z \phi \partial_{\bar{z}} \phi \,, \\ g_{zz} &= \int d^2x \partial_z \phi \partial_z \phi \,, \\ g_{\bar{z}\bar{z}} &= \int d^2x \partial_{\bar{z}} \phi \partial_{\bar{z}} \phi \,. \end{aligned} \tag{4.2}$$

It is now straightforward but somewhat tedious to plug in the function (3.6) for ϕ in these expressions and do the integrals to find the metric. We find a Kähler metric with Kähler potential $K(z,\bar{z}) = \ln \sinh(2z\bar{z})$.

In polar coordinates $\sqrt{2}z = re^{i\theta}$ the metric is $ds^2 = f(r)(dr^2 + r^2 d\theta^2)$ where

$$f = \coth\left(r^2\right) - \frac{r^2}{\sinh^2(r^2)} \,. \tag{4.3}$$

For large r, $f = 1$ and the metric becomes flat. This is expected, since the solitons do not feel each other when the separation is large, and hence the geodesics are just straight lines. For small r, the function f goes to zero as r^2. Introducing a new coordinate $\rho = r^2$ we obtain

$$ds^2 \propto r^2\left(dr^2 + r^2 d\theta^2\right) = \frac{1}{4}\left(d\rho^2 + 4\rho^2 d\theta^2\right) = \frac{1}{4}\left(d\rho^2 + \rho^2 d\tilde{\theta}^2\right) . \tag{4.4}$$

The last form of the metric looks flat, but the new coordinate $\tilde{\theta} = 2\theta$ takes values between 0 and 4π, and hence there is a conical singularity at the center of the moduli space.[2] For a cone, it is easy to compute the geodesics: in the $(\rho, \tilde{\theta})$ coordinates, the geodesics are straight lines, and thus a line coming in from $\tilde{\theta} = 0$ goes out at $\tilde{\theta} = \pi$. Consequently, in the physical coordinates (r, θ) a geodesic coming in from $\theta = 0$ goes out at $\theta = \pi/2$, and *all* scattering is right angle, independent of the impact parameter. In our case we expect to see this behavior only for geodesics that pass close to the singularity, corresponding to right angle scattering for small impact parameter (just as in the monopole case), whereas for large impact parameter we expect no scattering at all since the metric is flat for large separations as discussed above. In between we expect to see crossover behavior; we have checked our general arguments numerically.

[2]The metric (4.3) has appeared in a different physical context in [22, 23]; however, in that context the physics imposed different boundary conditions, and the space of [22, 23] is a $\mathbb{Z}_2$ orbifold of our space, and consequently is regular at $r = 0$ but approaches a cone with positive defect π for $r \to \infty$.

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We can solve the geodesic equation for this metric. The equation can be integrated to give

$$\frac{d\theta}{dr} = \pm \frac{1}{r\sqrt{(r/b)^2 f(r) - 1}}\,, \tag{4.5}$$

where b is an integration constant. The geodesic can then be found numerically as

$$\theta\,(r) = -\int_\infty^r \frac{ds}{s\sqrt{(s/b)^2 f(s) - 1}}\,. \tag{4.6}$$

For large r (where $f \approx 1$) the integral is elementary, and we find $\theta\,(r) \approx b/r$, which implies that b is the impact parameter. The exit angle (if the soliton comes in from $\theta = 0$) can similarly be found as

$$\theta_{\rm exit} = -2\int_\infty^{r_0} \frac{ds}{s\sqrt{(s/b)^2 f(s) - 1}}\,, \tag{4.7}$$

where we have introduced the constant r_0, the point of closest approach; this is related to the impact parameter by $b = r_0\sqrt{f\,(r_0)}$.

These formulas confirm our expectations about the scattering behavior.

5. Scattering — the general case

The extremely simple form of the moduli space metric in the previous section suggests that there should be a simple way to derive it and to show that it is Kähler. To show this we go over to the operator language, and use the correspondence

$$\int d^2x\ \partial_t\phi\partial_t\phi \longleftrightarrow {\rm Tr}(\dot{\mathcal{O}}\dot{\mathcal{O}})\,. \tag{5.1}$$

We will perform the derivation for the general case of scattering between solitons corresponding to the states $|n\rangle\langle n|$. In analogy with the previous section we define states $|n,\pm\rangle$ such that the two soliton solution $|n,+\rangle\langle n,+| + |n,-\rangle\langle n,-|$ at large separation look like two separated $|n\rangle\langle n|$ solitons. The $|n,\pm\rangle$ state will contain a piece proportional to ${\rm U}\,(z)\,|n\rangle \pm {\rm U}\,(-z)\,|n\rangle$ plus terms proportional to $|k\rangle$, $k < n$ which go to zero exponentially as $z \to \infty$. Thus, the operators we need are always of the form $|+\rangle\langle+| + |-\rangle\langle-|$ with $\langle+|-\rangle = 0$ and $\langle+|+\rangle = \langle-|-\rangle = 1$. In terms of these operators, the kinetic term from which we read off the metric is

$${\rm Tr}\left((|\dot{+}\rangle\langle+| + |+\rangle\langle\dot{+}| + |\dot{-}\rangle\langle-| + |-\rangle\langle\dot{-}|\,)^2\right), \tag{5.2}$$

which we can rewrite as

$$\begin{aligned} 2\,(&\langle\dot{+}|\dot{+}\rangle - \langle\dot{+}|+\rangle\langle+|\dot{+}\rangle - \langle\dot{+}|-\rangle\langle-|\dot{+}\rangle + \langle\dot{-}|\dot{-}\rangle - \langle\dot{-}|+\rangle\langle+|\dot{-}\rangle - \\ &- \langle\dot{-}|-\rangle\langle-|\dot{-}\rangle)\,, \end{aligned} \tag{5.3}$$

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since for any *constant* $\langle A|B\rangle$ we can "partially integrate" $\langle A|\dot{B}\rangle = -\langle\dot{A}|B\rangle$. As explained in [22, 23], this can be rewritten in a more compact form in terms of the projected derivatives

$$|D_t\pm\rangle \equiv |\dot{\pm}\rangle - |\pm\rangle\langle\pm|\dot{\pm}\rangle - |\mp\rangle\langle\mp|\dot{\pm}\rangle\,, \tag{5.4}$$

as

$$\langle D_t+|D_t+\rangle + \langle D_t-|D_t-\rangle\,. \tag{5.5}$$

Furthermore, in our case, we find that $\langle+|\dot{-}\rangle = \langle-|\dot{+}\rangle = 0$, which makes it possible to write the projected derivatives in a form that does not mix the $|+\rangle$ and $|-\rangle$ states

$$|D_t\pm\rangle = |\dot{\pm}\rangle - |\pm\rangle\langle\pm|\dot{\pm}\rangle\,, \tag{5.6}$$

and allows us to calculate the contributions from the $\pm$ states independently.

We calculate the metric at level n by a recursive/inductive procedure. We consider a state $|n,\pm\rangle$ that is normalized and orthogonal to all $|k,\pm\rangle, |k,\mp\rangle, k<n$, as well as to $|n,\mp\rangle$. The state has the form

$$|n,\pm\rangle = N_{n,\pm}\left[a^\dagger\frac{|n-1,\pm\rangle}{N_{n-1,\pm}} + \cdots\right] = N_{n,\pm}\left[(a^\dagger)^n\frac{|0,\pm\rangle}{N_{0,\pm}} + \cdots\right], \tag{5.7}$$

where $|0,\pm\rangle \equiv N_{0,\pm}(e^{za^\dagger} \pm e^{-za^\dagger})|0\rangle$ and the terms indicated by $\cdots$ are determined by the orthogonality conditions; we do not need their explicit form.[3] The metric at this level is computed from the z and $\bar{z}$ derivatives of $|n,\pm\rangle/N_{n,\pm}$:

$$|\partial n,\pm\rangle \equiv N_{n,\pm}\partial\left(\frac{|n,\pm\rangle}{N_{n,\pm}}\right), \tag{5.8}$$

$$|\bar{\partial} n,\pm\rangle \equiv N_{n,\pm}\bar{\partial}\left(\frac{|n,\pm\rangle}{N_{n,\pm}}\right). \tag{5.9}$$

The metric depends only on the projections of $|\partial n,\pm\rangle$ and $|\bar{\partial} n,\pm\rangle$ orthogonal to $|n,\pm\rangle$:

$$|D_z n,\pm\rangle \equiv |\partial n,\pm\rangle - |n,\pm\rangle\langle n,\pm|\partial n,\pm\rangle\,, \tag{5.10}$$

and similarly for $|\bar{\partial} n,\pm\rangle$; we shall shortly see that $|\bar{\partial} n,\pm\rangle$ is already orthogonal to $|n,\pm\rangle$, and hence $|D_{\bar{z}} n,\pm\rangle = |\bar{\partial} n,\pm\rangle$. We calculate the metric using

$$\begin{aligned}\frac{\partial}{\partial t}|n,\pm\rangle &= \left(\frac{\partial}{\partial t}N_{n,\pm}\right)\left(\frac{|n,\pm\rangle}{N_{n,\pm}}\right) + \dot{z}|\partial n,\pm\rangle + \dot{\bar{z}}|\bar{\partial} n,\pm\rangle \Rightarrow \\ |D_t n,\pm\rangle &= \dot{z}|D_z n,\pm\rangle + \dot{\bar{z}}|D_{\bar{z}} n,\pm\rangle\end{aligned} \tag{5.11}$$

and substituting into (5.5). Since, as we shall see,

$$\langle\partial n,\pm|D_z n,\pm\rangle = \langle D_{\bar{z}} n,\pm|\bar{\partial} n,\pm\rangle = 0\,, \tag{5.12}$$

[3] We have choosen the phase such that $N_{n,\pm}$ is always real.

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the metric is hermitean and takes the form

$$g_{z\bar{z}} = G_{n+} + G_{n-}\,, \tag{5.13}$$

where

$$G_{n,\pm} = \langle D_{\bar{z}} n, \pm | D_z n, \pm \rangle + \langle \partial n, \pm | \bar{\partial} n, \pm \rangle\,. \tag{5.14}$$

For future reference, we also define

$$g_{n,\pm} \equiv \langle D_{\bar{z}} n, \pm | D_z n, \pm \rangle\,. \tag{5.15}$$

We now determine $|D_z n, \pm\rangle$ and $|\bar{\partial} n, \pm\rangle$. We begin with the orthogonality relations

$$\begin{aligned} \langle k, \pm | n, \pm \rangle &= 0\,, \qquad k < n\,, \\ \langle k, \mp | n, \pm \rangle &= 0\,, \qquad k \leq n\,, \end{aligned} \tag{5.16}$$

and differentiate with respect to z and $\bar{z}$; we find

$$\begin{aligned} \langle \partial k \pm | n, \pm \rangle + \langle k, \pm | \partial n, \pm \rangle &= 0\,, \\ \langle \partial k \mp | n, \pm \rangle + \langle k, \mp | \partial n, \pm \rangle &= 0\,, \end{aligned} \tag{5.17}$$

$$\begin{aligned} \langle \bar{\partial} k \pm | n, \pm \rangle + \langle k, \pm | \bar{\partial} n, \pm \rangle &= 0\,, \\ \langle \bar{\partial} k \mp | n, \pm \rangle + \langle k, \mp | \bar{\partial} n, \pm \rangle &= 0\,. \end{aligned} \tag{5.18}$$

Our inductive assumption is

$$|\bar{\partial} k \pm\rangle \propto |k-1, \mp\rangle\,, \qquad k < n\,. \tag{5.19}$$

Then from eq. (5.17) we find that $|\partial n, \pm\rangle$ is orthogonal to all $|k, \pm\rangle$, $k < n$ and $|k, \mp\rangle$, $k \leq n$. Consequently, $|\partial n, \pm\rangle$ is a linear combination of $|n+1, \mp\rangle$ and $|n, \pm\rangle$. This allows us to use eq. (5.18) to determine $|\bar{\partial} n, \pm\rangle$:

$$|\bar{\partial} n, \pm\rangle = -\langle \bar{\partial} n - 1, \mp | n, \pm \rangle | n-1, \mp \rangle\,. \tag{5.20}$$

Because $|\partial n, \pm\rangle$ is a linear combination of $|n+1, \mp\rangle$ and $|n, \pm\rangle$, by construction, $|D_z n, \pm\rangle$ is proportional to $|n+1, \mp\rangle$. This implies

$$\begin{aligned} |\bar{\partial} n, \pm\rangle &= -\langle D_{\bar{z}} n - 1, \mp | n, \pm \rangle | n-1, \mp \rangle \\ &= -\sqrt{g_{n-1,\mp}}\, |n-1, \mp\rangle\,, \end{aligned} \tag{5.21}$$

where the last identity follows from eq. (5.15). Hence we have calculated part of the metric (5.14):

$$\langle \partial n, \pm | \bar{\partial} n, \pm \rangle = g_{n-1,\mp}\,, \tag{5.22}$$

and proven the inductive assumption (5.19) holds for $k = n$. We have also proven that $\langle n, \mp | \frac{\partial}{\partial t} | n, \pm \rangle = 0$, as required by (5.6), as well as the orthogonality constraints (5.12).

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We now calculate the norm of $|D_z n, \pm\rangle$ by differentiating $\langle n, \pm | n, \pm\rangle = 1$ with respect to z; using the definitions of the states $|\partial n, \pm\rangle, |\bar\partial n, \pm\rangle$ (5.8) and (5.9), and the orthogonality relation $\langle \partial n, \pm | n, \pm\rangle = 0$, we find

$$\langle n, \pm | \partial n, \pm\rangle + 2\partial \ln(N_{n,\pm}) = 0\,. \tag{5.23}$$

We may differentiate this relation with respect to $\bar z$ to obtain:

$$\begin{aligned}\langle \bar\partial n, \pm | \partial n, \pm\rangle + 2\bar\partial \ln(N_{n,\pm}) \langle n, \pm | \partial n, \pm\rangle + \\ + \langle n, \pm | N_{n,\pm} \bar\partial \left(\frac{|\partial n, \pm\rangle}{N_{n,\pm}} \right) + 2\partial\bar\partial \ln(N_{n,\pm}) = 0\,.\end{aligned} \tag{5.24}$$

Substituting (5.23) into (5.24) gives

$$\langle D_{\bar z} n, \pm | D_z n, \pm\rangle = -\langle n, \pm | N_{n,\pm} \bar\partial \left(\frac{|\partial n, \pm\rangle}{N_{n,\pm}} \right) - 2\partial\bar\partial \ln(N_{n,\pm})\,. \tag{5.25}$$

Differentiating the orthogonality relation $\langle n, \pm | \bar\partial n, \pm\rangle = 0$, we find

$$\langle \partial n, \pm | \bar\partial n, \pm\rangle + \langle n, \pm | N_{n,\pm} \partial \left(\frac{|\bar\partial n, \pm\rangle}{N_{n,\pm}} \right) = 0\,; \tag{5.26}$$

by the definitions of $|\partial n, \pm\rangle, |\bar\partial n, \pm\rangle$ (5.8) and (5.9), we have

$$\partial \left(\frac{|\bar\partial n, \pm\rangle}{N_{n,\pm}} \right) = \bar\partial \left(\frac{|\partial n, \pm\rangle}{N_{n,\pm}} \right), \tag{5.27}$$

and hence the norm of $|D_z n, \pm\rangle$ is (the square root of)

$$g_{n,\pm} = \langle \partial n, \pm | \bar\partial n, \pm\rangle - 2\partial\bar\partial \ln(N_{n,\pm})\,. \tag{5.28}$$

Consequently,

$$g_{n,\pm} = -\partial\bar\partial \ln(N^2_{n,\pm}) + g_{n-1,\mp} = -\partial\bar\partial \ln\left(N^2_{n,\pm} N^2_{n-1,\mp} \cdots N^2_{0,\pm(-1)^n} \right), \tag{5.29}$$

and the full metric obeys

$$G_{n,\pm} = -\partial\bar\partial \ln(N^2_{n,\pm}) + 2 g_{n-1,\mp}\,. \tag{5.30}$$

Clearly, this gives us an expression for the Kähler potential in terms of $N^2_{k,\pm}$ for $k \le n$.

To complete the determination of the metric, we need a formula for $N^2_{n,\pm}$; we calculate it, and find a second expression for the metric. From the definition of $|\partial n, \pm\rangle$ (5.8) and the orthogonality properties found above, we have

$$|D_z n, \pm\rangle = \frac{N_{n,\pm}}{N_{n,\mp}} \left(a^\dagger | n, \mp\rangle - |n, \pm\rangle \langle n, \pm | a^\dagger | n, \mp\rangle \right). \tag{5.31}$$

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To evaluate the second term, we need to find $a|n,\pm\rangle$. The leading term follows from the definition of $|n,\pm\rangle$ in eq. (5.7), and we are led to the ansatz

$$a|n,\pm\rangle = z\frac{N_{n,\pm}}{N_{n,\mp}}|n,\mp\rangle + C_{n,\pm}|n-1,\pm\rangle + \cdots\,, \tag{5.32}$$

where the terms represented by $\cdots$ involve $|k,\pm\rangle, |k,\mp\rangle$ for $k < n-1$ and will be shortly shown to vanish. Consider the inner products

$$\langle k,\pm|a|n,\pm\rangle, \qquad \langle k,\mp|a|n,\pm\rangle\,; \tag{5.33}$$

for $k < n-1$, since $a^\dagger|k,\pm\rangle \propto |k+1,\pm\rangle + \cdots$, the orthogonality properties of $|n,\pm\rangle$ immediately imply that all these inner products vanish. Hence, as promised above, the missing terms in (5.32) do indeed vanish. For $k = n-1$, we use

$$a^\dagger|n-1,\pm\rangle = \frac{N_{n-1,\pm}}{N_{n,\pm}}|n,\pm\rangle + \cdots\,. \tag{5.34}$$

Substituting this into (5.32) and (5.33), we find

$$C_{n,\pm} = \frac{N_{n-1,\pm}}{N_{n,\pm}}\,, \tag{5.35}$$

and hence

$$a|n,\pm\rangle = z\frac{N_{n,\pm}}{N_{n,\mp}}|n,\mp\rangle + \frac{N_{n-1,\pm}}{N_{n,\pm}}|n-1,\pm\rangle\,. \tag{5.36}$$

We thus find an explicit expression for $|D_z n,\pm\rangle$:

$$|D_z n,\pm\rangle = \frac{N_{n,\pm}}{N_{n,\mp}}\left(a^\dagger|n,\mp\rangle - \bar{z}\frac{N_{n,\pm}}{N_{n,\mp}}|n,\pm\rangle\right). \tag{5.37}$$

Because $|n,\pm\rangle \propto |D_z n-1\mp\rangle$, we find a recursive relation for the normalization factor $N_{n,\pm}$:

$$N_{n,\pm} = \frac{N_{n-1,\mp}}{\sqrt{g_{n-1,\mp}}}\,, \tag{5.38}$$

which allows us to calculate the metric recursively using (5.29). Alternatively, we can calculate the contribution (5.15) to the metric directly from (5.37):

$$g_{n,\pm} = \left(\frac{N_{n,\pm}}{N_{n,\mp}}\right)^2\left[1+\left(\frac{N_{n-1,\mp}}{N_{n,\mp}}\right)^2\right] + z\bar{z}\left[1-\left(\frac{N_{n,\pm}}{N_{n,\mp}}\right)^4\right]. \tag{5.39}$$

This bears no obvious resemblance to (5.29); we have explicitly verified the consistency of the formulas (5.39) and (5.29) to level $n = 3$.

Though the metrics increase in complexity as n increases, their qualitative behavior for large and small r is the same, and consequently, the scattering of higher n solitons should be basically the same as for the $n = 0$ case discussed in the previous section.

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6. Discussion

In [22, 23] it is described how the assumption that the state vectors are analytic functions of the complex moduli space coordinates, up to normalization factors, leads to a Kähler metric. Although this assumption does not hold for our case in this paper, we have shown that the metric on the relative moduli space of two solitons of the type $|n\rangle\langle n|$ for arbitrary n is nevertheless Kähler, and we have given a general expression for the Kähler potential of this metric. Our analysis was done for the case where the non-commutativity parameter $\theta \to \infty$. By finding the geodesics of this metric we studied the scattering of these solitons against each other and found a universal behavior in that the scattering always goes to right angle scattering for small values of the impact parameter. This we understood as coming from a conical singularity at the center of the moduli space.

A natural and important question is what happens at finite θ. We hope that the methods that we have developed in this paper can also be used in this case. Another important question where we should be able to use our methods is in theories with more non-commutative coordinates. One could also imagine studying non-spherically symmetric solutions by choosing a different U operator corresponding to squeezed states.

An equally interesting question is to take the viewpoint of [9, 10] where it was shown that the solitons treated in this paper can be seen as lower-dimensional D-branes of bosonic string theory. Following this logic what we have done in this paper is to study D-brane scattering. It would be interesting to compare our results with more direct calculations of these processes in the D-brane language.

It would also be interesting to investigate scattering of soliton solutions of other type of non-commutative theories. In particular it should be interesting to study the soliton solutions of non-commutative Yang-Mills theory in this context.

Note added, (in response to a question raised by the referee): In this paper we consider only moduli corresponding to translations. The symmetry group of the solutions is U(∞); this means that the moduli space is infinite dimensional. The motivation for our choice comes from considerations at finite θ, where the degeneracy along almost all of the moduli is lifted. Following [8] we can write the derivative terms that we dropped at infinite θ as

$$\frac{2\pi}{g^2}\,\mathrm{Tr}\left([a,A][A,a^\dagger]\right), \tag{6.1}$$

where A is an operator corresponding to the state in question. The contribution to the energy will in general depend on the moduli parameters and constitutes a potential for them. In the one soliton case, $A = U|n\rangle\langle n|U^\dagger$, the U that leads to a

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minimal (constant) contribution is a translation. It has (additional) energy

$$\frac{2\pi}{g^2}(2n+1)\,, \tag{6.2}$$

i.e. $2g^2/\pi$ for the lowest $n=0$ state. In the two soliton sector, where $A = |n+\rangle\langle n+| + |n-\rangle\langle n-|$, we again find independence of the moduli parameter. The energy contribution is

$$\frac{2\pi}{g^2}2(2n+1)\,, \tag{6.3}$$

i.e. $4g^2/\pi$ for the lowest $n=0$ state. Thus, the moduli spaces that we study for the lowest level remain degenerate to first order in perturbation theory, whereas the degeneracy is lifted for generic moduli. These and other aspects on multisoliton solutions will be discussed in a forthcoming paper [24].

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References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[4] I. Chepelev and R. Roiban, *Renormalization of quantum field theories on noncommutative R^d, 1. Scalars*, *J. High Energy Phys.* **05** (2000) 037 [hep-th/9911098].

[5] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B 376** (1996) 53.

[6] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072.

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[7] M. van Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, *J. High Energy Phys.* **03** (2000) 035 [hep-th/0002186].

[8] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[9] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[10] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as noncommutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

[11] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[12] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, *Phys. Lett.* **B 495** (2000) 407 [hep-th/0007043].

[13] D.P. Jatkar, G. Mandal and S.R. Wadia, *Nielsen-Olesen vortices in noncommutative abelian Higgs model*, *J. High Energy Phys.* **09** (2000) 018 [hep-th/0007078].

[14] D.J. Gross and N.A. Nekrasov, *Dynamics of strings in noncommutative gauge theory*, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204].

[15] R. Gopakumar, S. Minwalla and A. Strominger, *Symmetry restoration and tachyon condensation in open string theory*, hep-th/0007226.

[16] R.S. Ward, *A Yang-Mills-Higgs monopole of charge 2*, *Comm. Math. Phys.* **79** (1981) 317.

[17] J.P. Provost and G. Vallee, *Riemannian structure on manifolds of quantum states*, *Comm. Math. Phys.* **76** (1980) 289.

[18] N.S. Manton, *A remark on the scattering of BPS monopoles*, *Phys. Lett.* **B 110** (1982) 54.

[19] M.F. Atiyah and N.J. Hitchin, *Low-energy scattering of nonabelian monopoles*, *Phys. Lett.* **A 107** (1985) 21.

[20] T.M. Samols, *Vortex scattering*, *Comm. Math. Phys.* **145** (1992) 149.

[21] N.S. Manton, *Statistical mechanics of vortices*, *Nucl. Phys.* **B 400** (1993) 624.

[22] T.H. Hansson, S.B. Isakov, J.M. Leinaas, U. Lindström, *Classical phase space and statistical mechanics of identical particles*, quant-ph/0003121.

[23] T.H. Hansson, S.B. Isakov, J.M. Leinaas, U. Lindström, *Exclusion statistics in classical mechanics*, quant-ph/0004108.

[24] L. Hadasz, U. Lindström, M. Roček, R. von Unge, in preparation.

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